Spectral characterization of Poincaré-Einstein manifolds with infinity of positive Yamabe type

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SPECTRAL CHARACTERIZATION OF POINCARÉ-EINSTEIN MANIFOLDS WITH INFINITY OF POSITIVE YAMABE TYPE

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Abstract. In this paper, we give a sharp spectral characterization of conformally compact Einstein manifolds with conformal infinity of positive Yamabe type in dimension $n+1>3$. More precisely, we prove that the largest real scattering pole of a conformally compact Einstein manifold (X,g) is less than $\frac{n}{2}-1$ if and only if the conformal infinity of (X,g) is of positive Yamabe type. If this positivity is satisfied, we also show that the Green function of the fractional conformal Laplacian $P(\alpha)$ on the conformal infinity is non-negative for all $\alpha\in[0,2]$.

1. Introduction

Let Γ be a convex co-compact group without torsion of orientation preserving isometries of the $(n+1)$-dimensional real hyperbolic space $\mathbb{H}^{n+1}$, and let $\Omega(\Gamma)\subset S^n$ the domain of discontinuity of Γ. Then the hyperbolic manifold $X:=\Gamma\backslash\mathbb{H}^{n+1}$ is conformally compact with a conformal infinity M which is locally conformally flat and given by the compact quotient $M=\Gamma\backslash\Omega(\Gamma)$ when we view the elements of Γ as Möbius transformation acting on the closed unit ball of $\mathbb{R}^{n+1}$. In [23], Schoen and Yau proved that the Hausdorff dimension δ_Γ of the limit set $\Lambda(\Gamma)=S^n\setminus\Omega(\Gamma)$ of the group Γ is less than $\frac{n}{2}-1$ if the conformal infinity $\Gamma\backslash\Omega(\Gamma)$ is of positive Yamabe type (we say that a conformal manifold is of positive Yamabe type if and only if there is a Riemannian metric in its conformal class whose scalar curvature is positive). Later it was proved in [17] that the converse also holds. Sullivan [24] and Patterson [18] also proved that the Poincaré exponent of the group Γ is equal to δ_Γ. Moreover, in [20], Perry showed that the largest real scattering pole of $\Gamma\backslash\mathbb{H}^{n+1}$ is given by the Poincaré exponent $s=\delta(\Gamma)$ (see also [9] for a characterization of $\delta(\Gamma)$ in terms of first resonance). Therefore, in this context, we know that the largest real scattering pole of $\Gamma\backslash\mathbb{H}^{n+1}$ is less than $\frac{n}{2}-1$ if and only if the conformal infinity $\Gamma\backslash\Omega(\Gamma)$ is of positive Yamabe type. This result which relates the conformal geometry of the infinity $\Gamma\backslash\Omega(\Gamma)$ to the spectral property of the conformally compact hyperbolic manifold $\Gamma\backslash\mathbb{H}^{n+1}$ has been very intriguing.

Later in [12], Lee made a clever use of the positive generalized eigenfunctions to deduce that there is no L^2 eigenvalues in $(0,\frac{n^2}{4})$ on $(n+1)$-dimensional conformally compact Einstein manifolds X with conformal infinity of nonnegative Yamabe type. However, the particular case of hyperbolic convex co-compact quotients mentionned above shows that the absence of L^2 eigenvalues does not imply the positivity of the Yamabe type of the conformal infinity (the L^2-eigenvalues would be scattering poles in $(\frac{n}{2},n)$). A simple explicit example is just obtained by taking the quotient of $\mathbb{H}^3$ by a Fuchsian group Γ, giving rise to an infinite volume hyperbolic cylinder with section the Riemann surface $\Gamma\backslash\mathbb{H}^2$. In the introduction of [12], Lee asked what would be a sharp spectral condition for a conformally compact Einstein manifold to have a conformal infinity of positive Yamabe type. Considering the hyperbolic cases mentionned above, it is then natural to ask wether the fact that the largest real scattering pole is less than $\frac{n}{2}-1$ on conformally compact Einstein manifolds is equivalent to positivity of Yamabe type of the conformal infinity. In the spirit of the work of Lee [12], we are able to give such a spectral characterization of conformally compact Einstein manifolds with conformal infinity of positive Yamabe type.

Let us first introduce some notations and state our main theorem precisely. Suppose that X is an $(n+1)$-dimensional smooth manifold with boundary $\partial X = M$. A metric g on X is said to be *conformally compact* if, for a smooth defining function x of the boundary M in X, $x^2 g$ extends smoothly as a Riemannian metric to the closure $\bar{X}$. A conformally compact metric g is complete, has infinite volume, and induces naturally a conformal class of metrics $[\hat{g}] = [x^2 g|_{TM}]$ (here x ranges over the smooth boudary defining functions). As shown in [13], the sectional curvature of a conformally compact metric converges to $-|dx|^2_{x^2 g}$ when approaching the boundary M. Hence a metric g on X is naturally said to be *asymptotically hyperbolic* (AH in short) if it is conformally compact and the sectional curvatures converge to -1 at the boundary. A conformally compact Einstein manifold (X, g) is an AH manifold such that $\mathrm{Ric}(g) = -ng$.

If (X, g) is an AH manifold, we know (cf. [6, 4]) that for any representative $\hat{g} \in [\hat{g}]$, there is a unique geodesic defining function x of ∂X associated to the representative $\hat{g}$ such that the metric g has the geodesic normal form near the boundary

$$g = x^{-2}(dx^2 + g_x) \tag{1}$$

where g_x is a one-parameter smooth family of Riemannian metrics on M with $\hat{g} = \hat{g}$. In Mazzeo [13] and Mazzeo-Melrose [15], it is shown that the spectrum of the (non-negative) Laplacian Δ_g acting on functions on an AH manifold (X, g) consists of the union of a finite set $\sigma_p(\Delta_g) \subset (0, \frac{n^2}{4})$ of L^2-eigenvalues, and a half-line of continuous spectrum $[\frac{n^2}{4}, +\infty)$. Recently, Joshi-Sa Baretto [11] and Graham-Zworski [7] (building on [15, 10, 21]), introduced the scattering operators $S(s)$ on AH manifolds. For any $s \in \mathbb{C}$ such that

$$\mathrm{Re}(s) \geq \frac{n}{2}, \quad s(n-s) \notin \sigma_p(\Delta_g), \quad s \notin \frac{n}{2} + \frac{\mathbb{N}}{2},$$

and $f \in C^\infty(\partial X)$, there is a unique solution v to the equation

$$(\Delta_g - s(n-s))v = 0 \tag{2}$$

on X which can be decomposed as follows

$$v = F x^{n-s} + G x^s, \text{ with } F, G \in C^\infty(\bar{X}) \text{ and } F|_{\partial X} = f. \tag{3}$$

The scattering operator is the linear operator defined on $C^\infty(\partial X)$ by

$$S(s)f = G|_{x=0}. \tag{4}$$

If the metric g_x has an even Taylor expansion at $x = 0$ in powers of x, it is shown in [7] (see [8] for the analysis of the points in $(n+1)/2 - \mathbb{N}$) that $S(s)$ has a meromorphic continuation to the complex plane as a family of pseudo-differential operators of complex order $2s - n$ on ∂X. These results extend the analysis of [10, 19, 21] on hyperbolic manifold $\Gamma\backslash\mathbb{H}^{n+1}$ to the AH class. It is proved in [7] that $S(s)$ has first order poles at $\frac{n}{2} + \mathbb{N}$, the residues of which are the GJMS conformally covariant Laplacian on $(\partial X, [\hat{h}])$ constructed in [5] if g is asymptotically Einstein. For our purpose, it is more convenient to consider the renormalized scattering operator

$$P(\alpha) := 2^\alpha \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(-\frac{\alpha}{2})} S(\frac{n+\alpha}{2}). \tag{5}$$

Those $P(\alpha)$ at regular points are conformally covariant α-powers of the Laplacian, they are self-adjoint when α is real and unitary when $\mathrm{Re}(\alpha) = 0$, moreover $P(2)$ is the Yamabe operator of the boundary if the bulk space X is (asymptotically) Einstein. We thus call $P(\alpha)$ the *fractional conformal Laplacian* for obvious reason. The *first real scattering pole* is defined to be the largest real number s such that $\alpha = 2s - n$ is a pole of $P(\alpha)$.

Theorem 1.1. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$. The first real scattering pole is less than $\frac{n}{2} - 1$ if and only if its conformal infinity $(M, [\hat{g}])$ is of positive Yamabe type.*

We can also show that

Theorem 1.2. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$ with conformal infinity of positive Yamabe type. Then, for all $\alpha \in (0, 2]$, $P(\alpha)$ satisfies*
(a) *the first eigenvalue is positive;*
(b) *$P(\alpha)1$ is positive for any choice of representative $\hat{g}$ of the conformal infinity with positive scalar curvature;*
(c) *the first eigenspace is generated by a single positive function;*
(d) *its Green function is nonnegative.*

Remark 1.3. *1) In both cases, it will be clear from the proof that we actually only need to assume that*

$$\mathrm{Ric}(g) \geq -ng$$

and that g_x defined in (1) *has the asymptotic form near the boundary*

$$g_x = \hat{g} - \frac{2x^2}{n-2}\Big(\mathrm{Ric}(\hat{g}) - \frac{\hat{R}}{2(n-1)}\hat{g}\Big) + O(x^3)$$

where $\hat{g}$ is a metric on ∂X, $Ric(\hat{g})$ is its Ricci curvature tensor and $\hat{R}$ its scalar curvature. Metric with this asymptotic 'weakly Einstein' structure are discussed by Mazzeo-Pacard [16].
2) Although we do not discuss this here, the smoothness assumption of g_x up to the boundary

is not necessary, and a restricted smoothness assumption $C^{k,\alpha}(\bar{X})$ for some $k \geq 3$ could rather easily be obtained without much modification.
3) It is well known that those four properties in Theorem 1.2 all hold for the conformal Laplacian $P(2)$. However, when $\alpha \in (0,2)$, $P(\alpha)$ is a pseudo-differential operator (non-local) and it is interesting to see that these four properties continue to hold then.

Our proof is essentially based on the maximum principle and the existence of a positive supersolution for $\Delta_g - s(n-s)$. To construct this supersolution, we use a special boundary defining function constructed by Lee [12], which has the advantage of being a positive generalized (non L^2) eigenfunction. We shall recall some basic facts about conformally compact Einstein manifolds in the next section. Then in Section 3 we prove Theorem 1.1. Since the proof is rather simple we will carry out some basic calculations for the expansions of F for the convenience of the reader. Finally in Section 4 we prove Theorem 1.2. The crucial issue will be the nonnegativity of the Green function.

2. Positive generalized eigenfunctions

In this Section, we first lay out basic facts about conformally compact Einstein manifolds, then we recall the construction of positive generalized eigenfunctions, following [12, 1, 22]. Let (X, g) be a conformally compact Einstein manifold with conformal infinity $(M, [\hat{g}])$. Is is shown in [6, 4], that for any representative $\hat{g} \in [\hat{g}]$, there is a unique geodesic defining function x such that the metric has the geodesic normal form

$$g = x^{-2}(dx^2 + g_x) \tag{6}$$

near the boundary. Using this form and considering a Taylor expansion of g_x at $x = 0$, Einstein's equations turn into a system which can be solved asymptotically (see [2, 4]). One finds that, when n is odd, the metric has an expansion

$$g_x = \hat{g} + g^{(2)}x^2 + \text{ even powers in } x \ + g^{(n-1)}x^{n-1} + g^{(n)}x^n + O(x^{n+1}), \tag{7}$$

and, when n is even,

$$g_x = \hat{g} + g^{(2)}x^2 + \text{ even powers in } x \ + hx^n \log x + g^{(n)}x^n + O(x^{n+2}). \tag{8}$$

When n is odd, $g^{(2i)}$ for $2i < n$ are formally determined by the local geometry of $(M, \hat{g})$ and $g^{(n)}$ is trace free and nonlocal. When n is even, $g^{(2i)}$ for $2i < n$, h and the trace of $g^{(n)}$ are determined by the local geometry of $(M^n, \hat{g})$, h is trace free, and trace free part of $g^{(n)}$ is formally undetermined. Acually, for the purpose of this paper, we only need to assume that

$$g^{(2)} = -\frac{2}{n-2}\Big(\mathrm{Ric}(\hat{g}) - \frac{\hat{R}}{2(n-1)}\hat{g}\Big), \tag{9}$$

where $\mathrm{Ric}(\hat{g})$ is the Ricci curvature tensor of $\hat{g}$ and $\hat{R}$ is the scalar curvature of $\hat{g}$. The following positive generalized eigenfunction was first constructed and used by Lee [12]. Its importance in the results of [22, 1] is also worth mentioning. From Lemma 5.2 in [12], we have

Lemma 2.1. *Let (X, g) be a conformally compact Einstein manifold and assume that $\hat{g}$ is a representative in $[\hat{g}]$ of the conformal infinity $(M^n, [\hat{g}])$ and let x be the associated geodesic boundary defining function. Then there is a unique positive generalized eigenfunction u solving*

$$(\Delta_g + n + 1)u = 0$$

with expansion at the boundary

$$u = \frac{1}{x} + \frac{\hat{R}}{4n(n-1)}x + O(x^2). \tag{10}$$

The important observation by Lee [12] (see also an interesting interpretation of such observation in [22, 1]) is that the gradient of u is controlled by u:

Lemma 2.2. *Suppose that, in addition to the assumptions in Lemma 2.1, the scalar curvature satisfies $\hat{R} \geq 0$. Then one has*

$$|\nabla_g u|_g^2 < u^2 \text{ in } X \tag{11}$$

Proof. The proof is done in Proposition 4.2 of [12]. We repeat it for the the convenience of the reader. First the estimate near the boundary

$$u^2 - |\nabla_g u|^2 = \frac{\hat{R}}{n(n-1)} + o(1) \tag{12}$$

follows from the construction of generalized eigenfunctions in Graham-Zworski [7], then an easy computation using $\Delta_g u = -(n+1)u$ gives

$$\Delta_g(u^2 - |\nabla_g u|_g^2) = 2\langle(\mathrm{Ric}_g + n)du, du\rangle_g + 2\left|\frac{\Delta_g u}{n+1}g + \nabla_g^2 u\right|_g^2 \tag{13}$$

which is non-negative if $\mathrm{Ric}(g) \geq -ng$. From the strong maximum principle, $u^2 - |\nabla u|_g^2$ attains its minimum on ∂X and only on ∂X, or else is constant. But by (12), the minimum on ∂X is non-negative, so $|\nabla_g u|_g^2 \leq u^2$. If $u^2 - |\nabla_g u|^2$ is a positive constant, the proof is clearly finished, so it remains to show that u^2 can not be identically equal to $|\nabla_g u|^2$. If it were the case, an easy computation would give that, for $s > n/2$ and $\phi := u^{-s}$,

$$\Delta_g \phi = -s\phi\frac{\Delta_g u}{u} - s(s+1)\phi\frac{|\nabla_g u|^2}{u^2} = s(n-s)\phi$$

but since clearly $\phi \in L^2$, it contradicts the result of [12] showing that there is no L^2-eigenvalues in $(0, n^2/4)$. $\square$

We remark that for the above two lemmas to hold, we only need to assume that $\mathrm{Ric}(g) \geq -ng$ and the expansion (7) and (8) hold up to second order with $g^{(2)}$ given by (9).

3. Proof of Theorem 1.1

We present a proof of Theorem 1.1 in this section. First we restate the result of Lee [12] in terms of scattering pole as follows:

Theorem 3.1. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$ with conformal infinity of nonnegative Yamabe type. Then the first scattering pole is less than or equal to $\frac{n}{2}$.*

Here we used the identification of poles of $P(2s-n)$ and poles of the resolvent $R(s) := (\Delta_g - s(n-s))^{-1}$ in $\mathrm{Re}(s) > n/2$ (see for instance [19, Lemma 4.13]). Hence to push the first scattering pole down to $\frac{n}{2} - 1$, we first show that the scattering operator is regular at $\frac{n}{2}$. For this purpose, we review some of the spectral analysis on AH manifolds. By the result of Mazzeo-Melrose [15, 8], the resolvent of Laplacian $R(s)$ is bounded on $L^2(X)$ for

$$s \in \mathbb{C}, \quad \mathrm{Re}(s) > n/2, \quad s(n-s) \notin \sigma_p(\Delta_g),$$

and admits a meromorphic continuation to $\mathbb{C}$ as an operator mapping the space $\dot{C}^\infty(\bar{X})$ of smooth functions on $\bar{X}$ vanishing to infinite order at ∂X to the space $x^s C^\infty(\bar{X})$. Moreover the poles of $R(s)$, called *resonances*, are such that the polar part of the Laurent expansion of $R(s)$ is a finite rank operator. We first observe

Lemma 3.2. *The resolvent $R(s)$ is analytic at $\frac{n}{2}$ if and only if there is no function $v \in x^{\frac{n}{2}} C^\infty(\bar{X})$ such that $(\Delta_g - n^2/4)v = 0$.*

Proof. It is rather straightforward to see that Lemma 4.9 of Patterson-Perry [19] extends to our case, i.e. only a first order pole is possible for $R(s)$ at $\frac{n}{2}$. Indeed, by spectral theory $\frac{n}{2}$ can only be a pole of order at most 2. If it is of order 2, then $n^2/4$ is an L^2 eigenvalue for Δ_g and the coefficient of order $(s - \frac{n}{2})^{-2}$ is a finite rank projector on the L^2-eigenspace. The analysis of [15] (see the proof of Prop 3.3 in [8] for details) shows that the corresponding L^2 normalized eigenvectors $(v_k)_{k=1,\dots,K}$ would be in $x^{\frac{n}{2}} C^\infty(\bar{X})$, but to be in $L^2(X)$, this implies actually that $v_k \in x^{\frac{n}{2}+1} C^\infty(\bar{X})$ and by the indicial equation near ∂X,

$$(\Delta_g - n^2/4) x^j f(y) = -(j - n/2)^2 f(y) + O(x^{j+1}), \quad \forall f \in C^\infty(\partial X)$$

which implies $v_k = O(x^\infty)$. But Mazzeo's unique continuation theorem [14] shows that then $v_k = 0$ for all k. Then $\frac{n}{2}$ can only be a pole of order 1 of $R(s)$, in which case the residue of $R(s)$ is finite rank with range in $\ker(\Delta_g - \frac{n^2}{4}) \cap x^{\frac{n}{2}} C^\infty(\bar{X})$. Conversely assume that $R(s)$ is analytic at $\frac{n}{2}$ and that there is an $u \in x^{\frac{n}{2}} C^\infty(\bar{X})$ in $\ker(\Delta_g - n^2/4)$ with leading asymptotic $u \sim x^{\frac{n}{2}} f_0(y)$ as $x \to 0$. Then by Graham-Zworski [7], we can construct for a smooth family in $\mathrm{Re}(s) = n/2$ of solutions $u_s \in x^{n-s} C^\infty(\bar{X}) + x^s C^\infty(\bar{X})$ such that $u_{n/2} = u$,

$$(\Delta_g - s(n-s)) u_s = 0,$$

and

$$u_s = x^{n-s}(f_0 + x^2 z_s) + x^s (S(s) f_0 + x^2 w_s)$$

where $S(s)$ is the scattering operator, z_s, w_s are smooth functions on $\bar{X}$ depending smoothly on s on the line $\mathrm{Re}(s) = n/2$; notice from [7] that u_s can be taken to be of the form

$$u_s := x^{n-s}\Phi(s) - R(s)(\Delta_g - s(n-s))(x^{n-s}\Phi(s))$$

where $\Phi(s) \in C^\infty(\bar{X})$ is smooth in s on the line $\mathrm{Re}(s) = n/2$ and such that $x^{n/2}\Phi(n/2) = u$ and $(\Delta_g - s(n-s))(x^{n-s}\Phi(s)) = O(x^\infty)$. Since we assumed $R(s)$ analytic at $s = n/2$, then taking the limit as $s \to n/2$ gives $u = x^{\frac{n}{2}}(f_0 + S(n/2)f_0 + O(x^2))$, which implies that $S(n/2)f_0 = 0$, but this is not possible since $S(s)$ is unitary on the line $\mathrm{Re}(s) = n/2$ (for instance by Section 3 of [7]). Thus the proof is complete. □

To show that the resolvent at $\frac{n}{2}$ is analytic, we refine slightly Lee's argument.

Lemma 3.3. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$, with a conformal infinity of nonnegative Yamabe type. Let $k > 0$ and consider*

$$\phi = (ku)^{-\frac{n}{2}} \log(ku), \tag{14}$$

where u is the positive generalized eigenfunction in Lemma 2.1 associated with a choice of $\hat{g}$ of nonnegative scalar curvature. Then if k is chosen large enough, we have

$$\Delta_g \phi > \frac{n^2}{4}\phi \text{ in } X. \tag{15}$$

Proof. This is a simple calculation:

$$\begin{aligned}
\Delta_g \phi &= -\frac{n}{2}\phi\frac{\Delta_g u}{u} - \frac{n}{2}(\frac{n}{2}+1)\phi\frac{|\nabla_g u|_g^2}{u^2} + (ku)^{-\frac{n}{2}}((n+1)\frac{|\nabla_g u|_g^2}{u^2} + \frac{\Delta_g u}{u}) \\
&= \frac{n^2}{4}\phi + \frac{n(n+2)}{4}\phi(1 - \frac{|\nabla_g u|_g^2}{u^2}) - (n+1)(ku)^{-\frac{n}{2}}(1 - \frac{|\nabla_g u|_g^2}{u^2}) \\
&= \frac{n^2}{4}\phi + (ku)^{-\frac{n}{2}}(1 - \frac{|\nabla_g u|_g^2}{u^2})(\frac{n(n+2)}{4}\log(ku) - (n+1)) \\
&> \frac{n^2}{4}\phi,
\end{aligned}$$

in X provided

$$\log(ku) > \frac{4(n+1)}{n(n+2)}.$$

Here we have used (11) of the previous section. □

Theorem 3.4. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$, with conformal infinity of nonnegative Yamabe type. Then the resolvent $R(\lambda)$ is regular at $\frac{n}{2}$ and $S(\frac{n}{2}) = -\mathrm{Id}$.*

Proof. By Lemma 3.2, we simply need to prove that there is no nontrivial function v solving

$$(\Delta_g - \frac{n^2}{4})v = 0 \quad \text{in } X$$

with

$$v = Fx^{\frac{n}{2}}$$

for some smooth $F \in C^\infty(\bar{X})$. A straightforward computation gives

$$\begin{aligned} \Delta_g \frac{v}{\phi} &= \frac{\Delta_g v}{\phi} - 2\langle \nabla_g v, \nabla_g \frac{1}{\phi}\rangle_g - v\frac{\Delta_g \phi}{\phi^2} - v\frac{2|\nabla_g \phi|_g^2}{\phi^3} \\ &= -(\frac{\Delta_g \phi}{\phi} - \frac{n^2}{4})\frac{v}{\phi} + 2\frac{\nabla_g \phi}{\phi} \cdot \nabla_g \frac{v}{\phi}, \end{aligned} \tag{16}$$

where ϕ is defined in (14) in Lemma 3.3. Now, by considering the asymptotic behaviour of ϕ and v at the boundary, we easily see that

$$\frac{v}{\phi} \to 0$$

when approaching the boundary. Hence, if there is a negative interior minimum for v/ϕ at $p \in X$, the term $\nabla_g(v/\phi)$ vanishes at p in (16), but since $-((\Delta_g - n^2/4)\phi)/\phi > 0$ in X, we deduce that $\Delta_g(v/\phi)$ is positive near p, and this is not possible by applying the strong maximum principle in a small disc around p. We thus have

$$v \geq 0 \quad \text{on } X.$$

The same argument with an interior maximum shows that $v \leq 0$ and thus $v = 0$. To see $S(\frac{n}{2}) = -\mathrm{Id}$ in this case, the proof of Lemma 4.3 in [8] can be applied to our case mutatis mutandis: it shows that the scattering operator at $n/2$ is given by

$$S(n/2) = -(\mathrm{Id} - 2P_0)$$

where P_0 is a projector with respect to $L^2(M, \mathrm{dvol}_{\hat{g}})$ on the vector space

$$V := \{(x^{-\frac{n}{2}}u)|_{\partial\bar{X}} ; u \in \mathrm{Range}(\mathrm{Res}_{\frac{n}{2}} R(\lambda))\}.$$

In particular from Lemma 3.2, we obtain $S(\frac{n}{2}) = -\mathrm{Id}$. $\square$

So far we have improved Theorem 3.1 of Lee and obtained that the first scattering pole is less than $\frac{n}{2}$. To push further we need to show that the scattering operator $S(s)$ for all $s \in (\frac{n}{2}, \frac{n}{2}+1)$ has no kernel. Indeed, from the work of Joshi-Sa Barreto [11] (see [7, 19] for the constant curvature case), we know that $\tilde{P}(s) := (1+\Delta_{\hat{g}})^{-s/4}P(2s-n)(1+\Delta_{\hat{g}})^{-s/4}$ is a family of bounded Fredholm operators on $L^2(\partial\bar{X}, \mathrm{dvol}_{\hat{g}})$ and the theory of Gohberg-Sigal [3] can be used to deduce that, by the meromorphic functional equation (e.g. see section 3 in [7])

$$S(s)S(n-s) = \mathrm{Id},$$

the operator $\tilde{P}(2s-n)$ has a pole at $s_0 \in \{\mathrm{Re}(s) \leq n/2\}$ if and only if $\tilde{P}(n-2s_0)$ has a non-zero kernel, or equivalently $P(2s-n)$ has a pole at s_0 if and only if $\tilde{P}(n-2s_0)$ has non-zero kernel. Thus this corresponds to prove that for $s \in (\frac{n}{2}, \frac{n}{2}+1)$, there is no solution to the Poisson equation

$$(\Delta_g - s(n-s))v = 0 \quad \text{in } X \tag{17}$$

with

$$v \in x^{n-s} C^\infty(\bar{X}).$$

This can be compared to the result of Lee did [12]: he proved that there is no nontrivial solution to the same equation with

$$v = x^s F \text{ for some } F \in C^\infty(\bar{X})$$

and some $s \in (\frac{n}{2}, \frac{n}{2}+1)$. We now define the function

$$\psi := u^{-(n-s)}. \tag{18}$$

By (10), we have

$$\psi = x^{n-s} - \frac{(n-s)\hat{R}}{4n(n-1)} x^{n+2-s} + O(x^{n+2-s}). \tag{19}$$

It is also an easy calculation similar to (15) (see also [12] for the case $\psi = u^{-s}$) to see that for $s \in (\frac{n}{2}, \frac{n}{2}+1)$

$$\Delta_g \psi > s(n-s)\psi \quad \text{in } X. \tag{20}$$

In order to show that the kernel of $S(s)$ is 0 for $s \in (\frac{n}{2}, \frac{n}{2}+1)$, we need to find the second term in the expansion of $F \in C^\infty(\bar{X})$ at the boundary (recall $v = x^{n-s}F$ is a solution of (17)). This can be found for instance in [7], but we will give some details for the convenience of the reader since it is rather straightforward. Recall that, in the product decomposition $(0,\epsilon)_x \times M$ near the boundary, we have for any smooth function f defined on $(M, \hat{g})$ and any $z \in \mathbb{R}$

$$\begin{aligned}\Delta_g(fx^z) =& -\frac{fx^{n+1}}{\sqrt{\det g_x}} \partial_x (x^{1-n}\sqrt{\det g_x}\partial_x x^z) - \frac{x^{z+2}}{\sqrt{\det g_x}} \partial_\alpha (\sqrt{\det g_x} g_x^{\alpha\beta} \partial_\beta f) \\ =& z(n-z) f x^z - \frac{z}{2} f x^{z+1} \mathrm{Tr}_{\hat{g}}(\partial_x g_x) + x^{z+2}\Delta_{\hat{g}} f + o(x^{z+2}),\end{aligned} \tag{21}$$

where $\Delta_{\hat{g}}$ is the Laplacian of $(M, \hat{g})$. Hence, since

$$\mathrm{Tr}_{\hat{g}}(\partial_x g_x) = -\frac{\hat{R}}{(n-1)} x + O(x^3)$$

from (9), we have

$$(\Delta_g - s(n-s))(fx^{n-s}) = (\frac{(n-s)\hat{R}}{2(n-1)} f + \Delta_{\hat{g}} f) x^{n-s+2} + o(x^{n-s+2})$$

and

$$\begin{aligned}(\Delta_g - s(n-s))(hx^{n-s+2}) &= ((n-s+2)(s-2) - s(n-s)) h x^{n-s+2} + o(s^{n-s+2}) \\ &= -2(n+2-2s) h x^{n-s+2} + o(x^{n-s+2}).\end{aligned}$$

Therefore we have

$$F = f + \frac{1}{2(n+2-2s)}\Big(\frac{(n-s)\hat{R}f}{2(n-1)} + \Delta_{\hat{g}} f\Big)x^2 + o(x^2). \tag{22}$$

Lemma 3.5. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$, with conformal infinity of positive Yamabe type, and suppose that h is a solution to*

$$S(s)h = 0$$

on M for some $s \in (\frac{n}{2}, \frac{n}{2}+1)$. Then h must vanish on M.

Proof. First of all, the statement here is independent of the choice of representative in $[\hat{g}]$. We then choose a representative $\hat{g}$ whose scalar curvature is positive at every point on M. Assume that h is non identically 0, we may assume with no loss of generality that the maximum of h is 1 and is achieved at $p_0 \in M$. Then we consider the solution v to the Poisson equation

$$(\Delta_g - s(n-s))v = 0$$

on X with the expansion

$$v = Fx^{n-s} + Gx^s$$

where $F|_{x=0} = h$. Hence, combining (20) and the identity

$$\Delta_g \frac{v}{\psi} = -\Big(\frac{\Delta_g \psi}{\psi} - s(n-s)\Big)\frac{v}{\psi} + 2\frac{\nabla_g \psi}{\psi}\cdot \nabla_g \frac{v}{\psi}, \tag{23}$$

similar to (16), we deduce from the maximum principle (exactly like in the proof of Theorem 3.4) that v/ψ can not have an interior positive maximum in X. The function v/ψ extends continuously to $\bar{X}$ and since its maximum over the boundary is equal to 1, it is clear that $v \leq \psi$ on X. From (22), we have

$$v(x, p_0) = x^{n-s} + \frac{1}{2(n+2-2s)}\Big(\frac{(n-s)\hat{R}}{2(n-1)} + \Delta_{\hat{g}} h(p_0)\Big)x^{n-s+2} + o(x^{n-s+2}). \tag{24}$$

Recall that p_0 is a maximum point for h on M, which implies that $\Delta_{\hat{g}} h(p_0) \geq 0$. Comparing (19) and (24) near p_0, we obtain a contradiction with the fact that $v \leq \psi$. □

It is obvious that Theorem 3.4 and Lemma 3.5 imply that, for a conformally compact Einstein manifold with conformal infinity of positive Yamabe type, the first scattering pole is less than $\frac{n}{2} - 1$. On the other hand, if we know that the first scattering pole on an AH manifold is less than $\frac{n}{2} - 1$, then we have $P(0) = \mathrm{Id}$ and so the operator $P(\alpha)$ remains positive for all $\alpha \in [0, 2]$. In particular, the Yamabe operator $P(2)$ is positive and then it is well known that the conformal infinity is of positive Yamabe type. This achieves the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Statement (a) in Theorem 1.2 is a simple consequence of Theorem 1.1. Since

$$P(0) = \mathrm{Id}$$

and

$$P(2) = \Delta_{\hat{g}} + \frac{n-2}{4(n-1)}\hat{R}$$

both with positive first eigenvalue, and $P(\alpha)$ for $\alpha \in (0,2)$ has no kernel, the first eigenvalue of $P(\alpha)$ has to be positive for all $\alpha \in (0,2)$.

Statement (b) follows easily from the arguments used in the proof of Theorem 1.1. Let us give a short proof in the

Proposition 4.1. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$. Suppose that a representative $\hat{g}$ of the conformal infinity has positive scalar curvature on M. Then $P_{\hat{g}}(\alpha)1$ is positive for all $\alpha \in [0,2]$, where $P_{\hat{g}}$ denotes the operator $P(\alpha)$ defined using $\hat{g}$ for conformal representative in the conformal infinity.*

Proof. Let v be the solution to the Poisson equation

$$(\Delta_g - s(n-s))v = 0 \quad \text{in } X$$

with

$$v = Fx^{n-s} + Gx^s, \quad F, G \in C^\infty(\bar{X})$$

and expansions

$$F = 1 + \frac{(n-s)\hat{R}}{4(n+2-2s)(n-1)}x^2 + o(x^2), \quad G = S(s)1 + O(x^2), \tag{25}$$

where

$$\alpha = 2s - n \in (0,2).$$

Let ψ be the positive supersolution of $\Delta - s(n-s)$ defined in (18), then using (23), we derive from the maximum principle (exactly like in the proof of Theorem 1.1) that

$$v < \psi$$

in X. Then, from the expansion (19) and (25), we first conclude that $S(s)1$ has to be non-positive on M for $s \in (\frac{n}{2}, \frac{n}{2}+1)$ since $v - \psi = x^s S(s)1 + o(x^s)$. Now if $S(s)1$ vanishes at a point $p \in M$, we can consider again the asymptotics (19) and (25) along the line $\{y = p; x < \epsilon\}$ and by positivity of $\hat{R}(p)$ we obtain a contradiction with $v < \psi$ for x small enough. We thus conclude that $P_{\hat{g}}(\alpha)1 > 0$ everywhere on M for all $\alpha \in (0,2)$. On the other hand, $P_{\hat{g}}(\alpha)1 > 0$ holds at 0 and 2 obviously. This ends the proof. □

Though, for the differential operator $P(2)$, the positivity of the first eigenvalue implies the other three properties due to the maximum principle, it is not so straightforward for pseudo-differential operators like $P(\alpha)$ for $\alpha \in (0,2)$. Of course, the crucial issue is the nonnegativity of the Green function of the pseudo-differential operators $P(\alpha)$, or equivalently the non-positivity of the Green function of the scattering operator $S(s)$ for $s \in (\frac{n}{2}, \frac{n}{2}+1)$.

By [15], outside the diagonal the Schwartz kernel $R(s; m, m')$ of the resolvent $R(s) = (\Delta_g - s(n-s))^{-1}$ has the regularity

$$R(s; m, m') \in (xx')^s C^\infty(\bar{X} \times \bar{X} \setminus \mathrm{diag}_{\bar{X}}).$$

Consider the Eisenstein function $E(s) \in C^\infty(X \times \partial\bar{X})$ defined for $s \neq n/2$ and s not a pole of $R(s)$ by

$$E(s; m, y') := (2s-n)[x'^{-s} R(s; m, x', y')]_{x'=0}, \quad m \in X, y' \in \partial\bar{X}$$

it solves the equation (for all y' fixed in $\partial\bar{X}$)

$$(\Delta_g - s(n-s))E(s; \cdot, y') = 0 \quad \text{in } X.$$

From the structure of the resolvent above, we see that for y' fixed in $\partial\bar{X}$, the function $m \to E(s; m, y')$ is in $x^s C^\infty(\bar{X} \setminus \{y'\})$. Moreover (see [11] or [7]), the leading behavior of $E(s; x, y, y')$ as $x \to 0$ (and for $y \neq y'$) is given by

$$E(s; x, y, y') = x^s(S(s; y, y') + O(x))$$

where $S(s; y, y')$ is the Schwartz kernel of $S(s)$.

For $s \in (n/2, n/2+1)$ such that $S(s)$ is invertible, the Green kernel of $S(s)$ is given by $S(n-s; y, y')$ by the functional equation $S(s)S(n-s) = \mathrm{Id}$ (see again [7]). The behavior of $S(n-s; y, y')$ as $y \to y'$ is analyzed in [11] (see the Proof of Theorem 1.1 in [11] for the computation of the principal symbol of $S(s)$).

Lemma 4.1. *The leading asymptotic behavior of $S(s; y, y')$ at the diagonal is given by*

$$S(s; y, y') = \frac{\pi^{-\frac{n}{2}}\Gamma(s)}{\Gamma(s-\frac{n}{2})}(d_{\hat{g}}(y, y'))^{-2s} + O((d_{\hat{g}}(y, y'))^{-2s+1})$$

where $d_{\hat{g}}(\cdot, \cdot)$ denote the distance for the metric $\hat{g}$ on ∂X. In particular for $s \in (n/2, n/2+1)$, one has $\Gamma(n/2-s) < 0$ so $S(n-s; y, y')$ tends to $-\infty$ at the diagonal $\{y = y'\}$ of $\partial X \times \partial X$.

With the above understanding of the Green function $S(n-s; y, y')$ of the scattering operator $S(s)$ for $s \in (\frac{n}{2}, \frac{n}{2}+1)$ we know that the corresponding Eisenstein function $E(n-s)$ solves

$$(\Delta_g - s(n-s))E(n-s) = 0 \quad \text{in } X$$

with the expansion

(26)

$$\begin{aligned} E(n-s; x, y, y') = x^{n-s}\Big(&S(n-s; y, y') \\ &+ \frac{x^2}{2(n+2-2s)}\big(\frac{(n-s)\hat{R}}{2(n-1)}S(n-s; y, y') - \Delta_{\hat{g}}S(n-s; y, y')\big) \\ &+ o(x^2)\Big), \end{aligned}$$

near the boundary, $y \neq y'$, and where $y' \in \partial\bar{X}$ is fixed, when g is at least asymptotically Einstein up to the second order. Let us first deduce the following Lemmas, which will be useful later.

Lemma 4.2. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$ with conformal infinity of positive Yamabe type. Then the integral kernel $S(n-s; y, y')$ is non-positive for all $y, y' \in \partial\bar{X}$ and $s \in (\frac{n}{2}, \frac{n}{2}+1)$.*

Proof. The proof runs similarly to the proof of Lemma 3.5 except that $S(n-s; y, y')$ for a fixed $y' \in \partial\bar{X}$ and $s \in (\frac{n}{2}, \frac{n}{2}+1)$ is not bounded from below according to Lemma 4.1. □

Lemma 4.3. *Let $s \in (\frac{n}{2}, \frac{n}{2}-1)$, then for all fixed $y \in \partial X$, the set $\{y' \in \partial X; S(n-s; y, y') = 0\}$ has empty interior in ∂X.*

Proof. Assume $S(n-s; y, y') = 0$ for some fixed $y \in \partial X$ and y' in an open set $U \subset \partial X$, then by the indicial equation (21) we deduce easily that $E(n-s; x, y, y') = O(x^\infty)$ for $y \in U$ and by Mazzeo's unique continuation theorem [14] this would imply that $E(n-s; x, y, y') = 0$, which is not possible. □

As a consequence of Lemma 4.2 we have

Proposition 4.2. *Let (X, g) be a conformally compact Einstein manifold of dimension $n+1 > 3$, with conformal infinity of positive Yamabe type. Then, for each $\alpha \in (0, 2)$, the first eigenspace of $P(\alpha)$ is spanned by a single positive function.*

Proof. We first produce a positive eigenfunction for $P(\alpha)$ and $\alpha \in (0, 2)$. Since each $P(\alpha)$ for $\alpha \in (0, 2)$ is invertible and with nonnegative Green function given by $P(-\alpha)$ (thanks to the functional equation $P(\alpha)P(-\alpha) = \mathrm{Id}$) , we look for the eigenfunction of $P(-\alpha)$ as to maximize

$$\frac{\int_M f P(-\alpha) f \mathrm{dvol}_{\hat{g}}}{\int_M |f|^2 \mathrm{dvol}_{\hat{g}}}. \tag{27}$$

By Lemma 4.2, we know that

$$|P(-\alpha) f| \leq P(-\alpha)|f|, \tag{28}$$

hence

$$\frac{\int_M f P(-\alpha) f \mathrm{dvol}_{\hat{g}}}{\int_M |f|^2 \, \mathrm{dvol}_{\hat{g}}} \leq \frac{\int_M |f| P(-\alpha)|f| \mathrm{dvol}_{\hat{g}}}{\int_M |f|^2 \mathrm{dvol}_{\hat{g}}}.$$

Therefore there is a nonnegative function $f \geq 0$ which is the first eigenfunction

$$P(\alpha) f = \lambda(\alpha) f. \tag{29}$$

It is then easily seen that f has to be positive, again due to Lemma 4.2. Namely, if $f(y) = 0$ and $P(-\alpha; y, y')$ is the Green function of $P(\alpha)$, then,

$$0 = f(y) = \lambda(\alpha) \int_M P(-\alpha; y, y') f(y') \mathrm{dvol}_{\hat{g}}(y'),$$

which implies $f \equiv 0$. Next we show that, if h is another eigenfunction of $P(\alpha)$ with eigenvalue $\lambda(\alpha)$, then the ratio $\frac{h}{f}$ has to be a constant on M. We shall use the conformal covariance property of the regularized scattering operator. Let us denote $P_{e^{2\omega}\hat{g}}(\alpha)$ the operator $P(\alpha)$ defined using the conformal representative $e^{2\omega}\hat{g} \in [\hat{g}]$ instead of $\hat{g}$, or equivalently using the boundary defining function $e^{\omega}x$. Then we have by the conformal covariance of $P(\alpha)$

$$P_{u^{\frac{4}{n-\alpha}}\hat{g}}(\alpha) = u^{-\frac{n+\alpha}{n-\alpha}} P_{\hat{g}}(\alpha) u, \tag{30}$$

for any positive function u on M. Hence

$$P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)\frac{h}{f} = f^{-\frac{n+\alpha}{n-\alpha}} P_{\hat{g}}(\alpha) h = f^{-\frac{n+\alpha}{n-\alpha}} \lambda(\alpha) h = f^{-\frac{n+\alpha}{n-\alpha}} (P_{\hat{g}}(\alpha) f) \cdot \frac{h}{f} = (P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha) 1) \cdot \frac{h}{f}, \tag{31}$$

where

$$P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)1 = \lambda(\alpha) f^{-\frac{2\alpha}{n-\alpha}} > 0.$$

Let $P_{f^{\frac{4}{n-\alpha}}\hat{g}}(-\alpha; y, y') \geq 0$ be the Green function of $P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)$. Then

$$\frac{h}{f}(y) = \int_M P_{f^{\frac{4}{n-\alpha}}\hat{g}}(-\alpha; y, y')((P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)1) \cdot \frac{h}{f})(y') \mathrm{dvol}_{f^{\frac{4}{n-\alpha}}\hat{g}}(y'). \tag{32}$$

Using that

$$\int_M P_{f^{\frac{4}{n-\alpha}}\hat{g}}(-\alpha; y, y')(P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)1)(y') \mathrm{dvol}_{f^{\frac{4}{n-\alpha}}\hat{g}}(y') = 1,$$

we deduce from (32)

$$0 = \int_M P_{f^{\frac{4}{n-\alpha}}\hat{g}}(-\alpha; y, y')(P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)1)(y') \Big[\frac{h}{f}(y) - \frac{h}{f}(y')\Big] \mathrm{dvol}_{f^{\frac{4}{n-\alpha}}\hat{g}}(y').$$

Since the Green kernel $P_{f^{\frac{4}{n-\alpha}}\hat{g}}(-\alpha; y, y')$ and $(P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)1)(y')$ are respectively non-negative and positive by (b) and (d) of Theorem 1.1, we deduce that for all $y \in \partial X$, $\frac{h}{f}(y) = \frac{h}{f}(y')$ for all $y' \neq y$ such that $P_{f^{\frac{4}{n-\alpha}}\hat{g}}(-\alpha; y, y')(P_{f^{\frac{4}{n-\alpha}}\hat{g}}(\alpha)1)(y') \neq 0$. But from Lemma 4.3, we know that for each y, this set is dense in ∂X. By continuity of h and f (which follows from ellipticity of $P(\alpha)$), we can conclude that $h = f$. Thus the proof is complete. $\square$

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